Continuation Value is All You Need

Simple Deep Learning for HA Models with Aggregate Shocks

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Overview

1. Describe model class and algorithm
2. Contrast with literature
3. Discuss performance, flexibility, and accuracy
4. Krusell-Smith benchmark

Algorithm

Model Class

- Discrete time. Het. atomistic agents. Beginning of period aggregate state:

$$\Gamma \equiv (\underbrace{\Lambda}_{\text{Household state distribution}}, \underbrace{S}_{\text{Other aggregate state variables}}).$$

- Timing within period:
    1. Beginning of period: Household with state $x \in X$ has value $V^{\text{start}}(x; \Gamma)$
    2. End of (household) period: Households have value $V^{\text{end}}(x; \Gamma)$ and distribution Λ^{end}
    3. Aggregate shock: $\Gamma' = \Omega\big(\Gamma, \Lambda^{\text{end}}, \varepsilon\big)$, with $\varepsilon \sim \text{Cat}(\{p_i\})$
- Key assumption: $V^{\text{start}}|_\Gamma : X \to \mathbb{R}$ is a function of $V^{\text{end}}|_\Gamma : X \to \mathbb{R}$ and prices[1]

[1] "prices" $\equiv$ some set of equilibrium values

Data Representation

$$\text{Fix some } \Gamma \equiv (\underbrace{\Lambda}_{\text{Household state distribution}}, \underbrace{S}_{\text{Other aggregate state variables}})$$

- Represent $V^{\text{start}}|_\Gamma$, $\Lambda^{\text{start}}|_\Gamma$ by data arrays $A_\Gamma^{V\text{start}}$, $A_\Gamma^{\Lambda\text{start}}$. Similarly, $A_\Gamma^{V\text{end}}$, $A_\Gamma^{\Lambda\text{end}}$.
- Define the **Intra-Period Problem** function mapping V backward and Λ forward:

$$\text{IPP} : \left(A_\Gamma^{V\text{end}}, A_\Gamma^{\Lambda\text{start}}, S_\Gamma\right) \mapsto \left(A_\Gamma^{V\text{start}}, A_\Gamma^{\Lambda\text{end}}\right)$$

- IPP can typically be implemented *conventionally* (no neural net)
- To truly solve a model, the Hard Part is knowing $A_\Gamma^{V\text{end}}$
- Strategy: Train a neural net $\mathcal{N}(\Gamma;\theta)$ to approximate $A_\Gamma^{V\text{end}}$
  - $\mathcal{N}$ uses "generalized moments" of Han et al. (2024)

Algorithm for Continuation Value Neural Net (One-Period Lookahead)

1. Guess neural net parameters θ_0
2. For each epoch $i \in \{1, \ldots, I\}$, simulate the model given $\mathcal{N}(\,\cdot\,; \theta_i)$, then update $\mathcal{N}$:
    - 2.1 Initialize state $\Gamma_{i0} \equiv \left(A_{i0}^{\Lambda\text{start}}, S_{i0}\right)$
    - 2.2 For each period $t \in \{1, \ldots, T\}$:
        - 2.2.1 Approximate end-of-period value array $A_{it}^{V\text{end}} \leftarrow \mathcal{N}(\Gamma_{it}\ ;\ \theta_i)$
        - 2.2.2 Compute $\left(A_{it}^{V\text{start}}, A_{it}^{\Lambda\text{end}}\right) \leftarrow \text{IPP}\left(A_{it}^{V\text{end}}, A_{it}^{\Lambda\text{start}}, S_{it}\right)$
        - 2.2.3 Draw $\varepsilon_{it} \sim \text{Cat}(\{p_i\})$
        - 2.2.4 Iterate state $\Gamma_{i,t+1} \leftarrow \Omega\left(A_{it}^{\Lambda\text{end}}, \Gamma_{it}, \varepsilon_{it}\right)$
    - 2.3 Update $\theta_i \rightarrow \theta_{i+1}$ with cost function, for sample periods $\mathcal{T}_i \subseteq \{1, \ldots, T\}$:

$$\frac{1}{|\mathcal{T}_i|} \sum_{t \in \mathcal{T}_i} \left| A_{it}^{V\text{end}} - \frac{1}{K} \sum_{k=1}^{K} p_k \widehat{A_{i,t+1}^{V\text{start}}}(\Gamma_{it}, \theta_i \mid \varepsilon_{it} = k) \right|^2$$

Details

Literature

Key choices (within HA model solutions w/ aggregate shocks)

- Time:
  - **Discrete**
  - Continuous: Gu et al. (2024), Fernández-Villaverde et al. (2023), etc.
- Solution scope
  - **Global**: Most DL based methods
  - Local: Most projection/perturbation methods: Bhandari et al. (2023), Bilal (2023), Auclert et al. (2021), Winberry (2018), etc.
- Policy function
  - **Conventional**: Krusell and Smith (1998), Hull (2015), etc.
  - Deep Learning: Han et al. (2024), Azinovic et al. (2022), Maliar et al. (2021), etc.
- Household simulation (all compatible with continuation value strategy):
  - Discrete State: Gu et al. (2024) do both, Kaplan et al. (2020)
  - Finite Agent: Krusell and Smith (1998), Han et al. (2024), etc.
  - Personal preference: **Gridded CDF**: fast, deterministic, less biased than point-mass

Classification

- Krusell-Smith/Hull with NN V and generalized moments
- Q-learning with equilibrium
- Approximate Dynamic Programming with Post-Decision States (ADP-POST) (Powell, 2007) with deep learning and equilibrium

Discussion

Flexibility

Key advantage: No need to train a policy function approximator, often the hardest part

- Only need to implement

$$\text{IPP}: \left(A_{\Gamma}^{V\text{end}}, A_{\Gamma}^{\Lambda\text{start}}, S_{\Gamma}\right) \to \left(A_{\Gamma}^{V,\text{start}}, A_{\Gamma}^{\Lambda\text{end}}\right)$$

- Can typically be done conventionally. Immediately correct as a function of $A_{\Gamma}^{\Lambda,\text{end}}$

I provide code to modularly implement IPP for household problems featuring:

- Consumption-saving decisions
- Idiosyncratic income shocks
- Binding borrowing constraints
- Multiple assets
- Multiple locations and frictional migration
- Real estate markets
- All of the above simultaneously

Performance

$$\text{IPP} : \left(A_\Gamma^{V,\text{end}}, A_\Gamma^{\Lambda,\text{start}}, S_\Gamma\right) \to \left(A_\Gamma^{V,\text{start}}, A_\Gamma^{\Lambda,\text{end}}\right)$$

- IPP function is inner loop of algorithm: costly need for many simulations
- IPP represents the bottleneck, but also the target for optimization
- The available code for building IPP functions is highly optimized and reusable
- Ideally, one person can contribute a new IPP module (or "stage"), many can use
- Cannot handle high-dimensional *individual* state

Accuracy

Suppose IPP is implemented:

$$\text{IPP} : \left(A_\Gamma^{V,\text{end}}, A_\Gamma^{\Lambda,\text{start}}, S_\Gamma\right) \mapsto \left(A_\Gamma^{V,\text{start}}, A_\Gamma^{\Lambda,\text{end}}\right).$$

- Immediately correct as a function of $A_\Gamma^{\Lambda,\text{end}}$. Only V needs to be trained
- Formally, IPP represents the solution to a one-period model where $A_\Gamma^{\Lambda,\text{end}}$ represents terminal payoffs
- If prices cannot be solved analytically, two options:
    1. Solve by inner loop around each IPP (slow but accurate)
    2. Introduce new price approximator neural net à la Azinovic et al. (2022)

Benchmark: Krusell-Smith Model

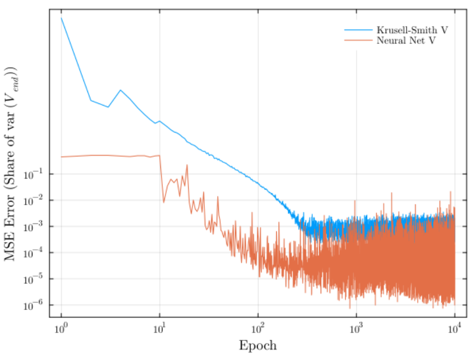
Model Setup

- Standard Krusell-Smith Model Details
- 65 wealth gridpoints, 3 income gridpoints = 195 idiosyncratic gridpoints
- Identical simulate-update_V loop, except V can either be parameterized as:
  1. Neural network Details
  2. Interpolation over k-l-K-A grid Details
- Error function:

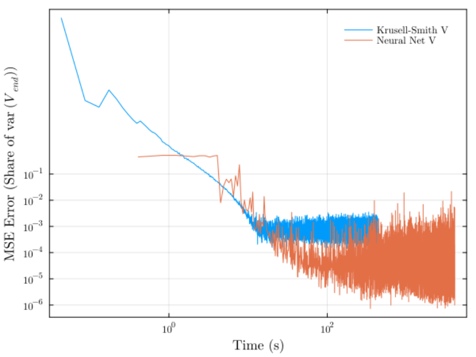
$$\frac{1}{|\mathcal{T}_i|}\sum_{t\in\mathcal{T}_i}\left|A_{it}^{V\text{end}} - \frac{1}{K}\sum_{k=1}^{K} p_k \widehat{A_{i,t+1}^{V\text{start}}}(\Gamma_{it}, \theta_i \mid \varepsilon_{it} = k)\right|^2$$

- Report: V error as share of var(V) across population
- Hardware: One laptop CPU thread (i9-13900H)

Learning Curve Comparison



V error as share of V variance by training epoch



V error as share of V variance by training time

- Both stop improving within about 300 epochs (122s for NN, 13s for KS)
- After Epoch 300, mean $\log_{10}$ error is -4.431 for NN, -3.152 for KS

Conclusion

Conclusion

- Describe a solution method for HA models with agg. shocks that overcomes curse of dimensionality (of aggregate state)
- Method uses neural nets only where needed – solution otherwise conventional
- Much complexity is offloaded to **Intra-Period Problem** (IPP) function
- In other work, provide tools to implement IPP flexibly, easily, performantly

Discussion

- Key advantages:
  - Complex household problems supported
  - No need to train policy network
- Disadvantages:
  - Individual state must be low-dimensional (≤ 6 or so)
  - Prices require inner loop around IPP or price neural net à la Azinovic et al. (2022)
- Future work:
  - Train to tighter tolerance
  - Assess other error metrics, e.g. Euler equation error
  - Compare economics of solutions

Appendix

Algorithm Details

- I use a gridded CDF representation of Λ, but using a finite number of agents is also possible. However, they have to interpolate over continuation V
- Training data for each simulated period is $A_{it}^{V\text{end}}$ together with

$$\mathbb{E}\left[A_{i,t+1}^{V,\text{start}} \mid \Gamma_{it}\right] = \frac{1}{K}\sum_{k=1}^{K} \text{IPP}_V\left(\mathcal{N}(\Omega(\Gamma_{it},k)\ ;\ \theta_i), A_{it}^{\Lambda\text{end}}, \Omega(\Gamma_{it},k)\right)$$

- For large models, if memory is constrained, you can update θ_i as you go, accumulating gradients but not storing the entire simulation

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Neural Network Implementation

Neural net $\mathcal{N}(A_{\Gamma}^{\Lambda\text{start}} ; \theta)$ has following components:

1. Generalized-moment of Han et al. (2023):

$$\text{GM}_{\Gamma} = \sum_{j \in J} \left(A_{\Gamma}^{\Lambda\text{start}}\right)_j \mathcal{N}_{\text{GM}}(X_j)$$

2. One layer $(1 \Rightarrow 10)$ neural net on aggregate productivity A
3. Dense feedforward neural net on input: $(X_j, \text{GM}_{\Gamma}, \mathcal{N}_A(A))$
    - Three hidden layers with 8, 8, and 5 neurons
    - Elu activation

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Krusell-Smith Model Details

- 65 wealth gridpoints
- 3 income gridpoints
- $\beta = 0.98$
- Income process by Tauchen discretization with persistence 0.95 and std 0.1
- Log-linear wealth grid from 1k to 10m
- Income states: 15.4k, 40.3k, 105.4k
- Risk aversion: 0.9
- Capital share: 0.36
- Depreciation rate: 0.025

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Krusell-Smith Method Details

- 5 aggregate capital gridpoints: 100k, 150k, 200k, 250k, 300k
- Linear extrapolation outside aggregate capital grid
- 2 aggregate productivity states: 0.5 and 1.0
- Unlike Krusell and Smith (1998), use gridded CDF population distribution representation for cleaner comparison

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